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On the range of applicability of Bourret approximation

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In the analysis of many dynamic soil-foundation interaction problems we have to deal with the unbounded dynamic systems for which the material parameters of the semi-infinite elastic medium are random functions of space variables. This paper, related to the above-mentioned problem deals with the range of validity of the approximate average solution for the simplified equation of motion. The solution is obtained by means of two different methods: Adomian's decomposition and Bourret's approximation. The formula for the error evaluation is derived and mean square convergence of the solution series is proved. Parametric study shows that Bourret's approximation may lead to quite reasonable results.

Keywords: stochastic waves, random oscillator, stochastic systems

1. Introduction

The reduced equation of motion for time-harmonic wave propagation in a randomly inhomogeneous semi-infinite elastic medium can be represented in the form

$$\frac{d\mathbf{u}}{dx} + \mathbf{A}\mathbf{u} = \mathbf{g}[\mathbf{u}, \boldsymbol{\varepsilon}(x,\gamma)], \qquad x \geqslant 0 \tag{1}$$

The different forms of equation (1) with specified initial and/or boundary conditions have attracted considerable attention and can be found in many papers.[1–5]

In equation (1) $\mathbf{u}$ is a displacement vector, a matrix $\mathbf{A}$ consists of deterministic parameters related to the properties of the medium, components of a vector $\boldsymbol{\varepsilon}(x,\gamma)$ represent random fluctuations of the material parameters of the medium, x-axis is directed to the interior of the semi-infinite medium, $\gamma \in \Gamma$ is the elementary event in a complete probability space (Γ, F, P) with F, as a σ-algebra of subsets of Γ, and P is a probability measure defined on all F, which induces suitable statistical measures. In Refs. [1–3] the system of equation (1) has been solved in the range of correlation theory for the average displacements on the basis of Bourret approximation.[4] The method used there does not give possibility of analyzing the error introduced by such approximation. Hence this method has been severely criticized.[5]

It has been concluded in many papers that Adomian's decomposition procedure[6] gives the possibility of evaluating the error of the approximate solution and of obtaining the solution with the assumed level of accuracy.

Adomian's method raises several mathematical aspects: Convergence problem of the series, accuracy of the approximate average solution, and accuracy of the variance function. The answer to the above questions has to be studied for each particular case, and it can be found when the structure of the operator is specified. Using Bourret's approximation allow us to obtain the solution in a simpler way, and it leads to computationally more efficient procedures.

Avoiding remaining at a formal level we specify the important case of equation (1) with initial conditions. The mean square convergence of the solution series is proved, and the formula for the evaluation of the error associated with various order approximations of the average solution is derived.

2. Problem formulation

The simple case of equation (1) with the initial conditions can be written in dimensionless form as

$$\frac{d^2 w}{dx^2} + w = -\varepsilon(x,\gamma) w, \qquad x \geqslant 0 \tag{2}$$

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$$w(0) = 1 \tag{3a}$$

$$\left.\frac{dw}{dx}\right|_{x=0} = 0 \tag{3b}$$

where x and w can be treated as a dimensionless space variable and displacement, respectively, $\varepsilon(x,\gamma)$ is a random function describing the randomness of the semi-infinite elastic medium and

$$\langle \varepsilon(x,\gamma) \rangle = 0 \tag{4}$$

The brackets $\langle . \rangle$ in this paper denote the averaging. Assuming $\varepsilon(x,\gamma)$ is a homogeneous random function,[4] one can describe it in the range of correlation theory by means of the average (equation [4]) and correlation function

$$\langle \varepsilon(x,\gamma)\varepsilon(x_1,\gamma) \rangle = K(x - x_1) \tag{5}$$

The wide class of the correlation functions may be represented in the form of the spectral density as[7]

$$S_p(\omega) = \frac{(2b)^{2n-1}[\Gamma(n)]^2 s^2}{(2n-2)!(\omega^2 + b^2)^n} \tag{6a}$$

In further considerations we will use case (6a) for $n = 1$.

While for $n = 1$ the assumed exponential correlation function has the undesirable mathematical property that the associated random function is not mean square differentiable it has been used in a number of investigations because it fits the experimental data.[1] To avoid this undesirable property one can substitute $n = 3$ in equation (6a), obtaining two times the differentiable associated random function.

It should be noted that if the stochastic characterization of $\varepsilon(x,\gamma)$ is motivated by uncertainty due to lack of information, resulting from a limitation in the density of field testing, the type of model given as

$$S_p(\omega) = \begin{cases} \dfrac{s^2}{\pi b}, & |\omega| \leq \dfrac{\pi b}{2} \\ 0, & |\omega| > \dfrac{\pi b}{2} \end{cases} \tag{6b}$$

seems also to be reasonable.[8]

Equation (2) can be represented in classical "mild" integral form

$$w = Y_1 G'(x) + Y_2 G(x) - \int_{x_1=0}^{x} G(x - x_1)\varepsilon(x_2)w(x_1)\,dx_1 \tag{7}$$

where $G(x)$ is a one-sided Green's function for the deterministic operator $d^2/dx^2 + 1$, and Y_1 and Y_2 are temporary initial conditions, which are viewed as integration constants.

In this paper the prime "$'$" on the dependent variable denotes derivative ($G'(x) = dG/dx$), and for the sake of conciseness the probability variable γ will be dropped.

The problem addressed here is that of determining the mean field $\langle w \rangle$ for equations (2) and (3). The parametric study, which has been done in another part of this paper, shows that Bourret's approximation

$$\langle \varepsilon(x)\varepsilon(x_1)w(x_1) \rangle \approx K(x - x_1)\langle w(x_1) \rangle \tag{8}$$

may be used in considering several still interesting physical problems.[9,10] Due to the fact that this method does not allow evaluation of the error of the approximate solution it is necessary to compare it with the solution obtained on the basis of another method.

3. Adomian's decomposition

Assume that the unknown random function $w(x)$ in equations (2) and (3) can be represented as a sum of the undefined deterministic function $w_0(x)$ and random functions $w_n(x)$, $(n = 1, 2, \ldots)$

$$w(x) = w_0(x) + \sum_{n=1}^{\infty} w_n(x) \tag{9}$$

Substituting equation (9) in equation (7), employing initial conditions, equation (3), and equating terms of the same order yields

$$w_0(x) = G'(x) \tag{10a}$$

$$w_1(x) = -\int_{x_1=0}^{x} G(x - x_1)\varepsilon(x_1)w_0(x_1)\,dx_1 \tag{10b}$$

$$w_n(x) = -\int_{x_1=0}^{x} G(x - x_1)\varepsilon(x_1)w_{n-1}(x_1)\,dx_1, \quad (n = 2, 3, \ldots) \tag{10c}$$

Back-substitution of the terms $w_n(x)$ in equation (10) leads to

$$w_1(x) = -\int_{x_1=0}^{x} G(x - x_1)G'(x_1)\varepsilon(x_1)dx_1 \tag{11a}$$

$$w_n(x) = (-1)^n \int_{x_1=0}^{x} \cdots \int_{x_n=0}^{x_{n-1}} [G(x - x_1)\ldots G(x_{n-1} - x_n)] G'(x_n)[\varepsilon(x_1)\ldots\varepsilon(x_n)][dx_n \ldots dx_1], \quad (n = 2, 3, \ldots) \tag{11b}$$

Utilizing the form of the general term in equation (11) and confining considerations to the range of the correlation theory one obtains the second-order approximation for the average solution as

$$\langle w(x) \rangle \approx \langle w^{(2)}(x) \rangle = w_0(x) + \langle w_2(x) \rangle \tag{12}$$

where

$$\langle w_2(x)\rangle = \int_{x_1=0}^{x}\int_{x_2=0}^{x_1} G(x-x_1)G(x_1-x_2)G'(x_2) \times K(x_1-x_2)\,dx_2\,dx_1 \quad (13)$$

The fourth-order approximation for the average solution assumes the form

$$\langle w(x)\rangle \approx \langle w^{(4)}(x)\rangle = w_0(x) + \langle w_2(x)\rangle + \langle w_4(x)\rangle \quad (14)$$

where

$$\langle w_4(x)\rangle = \int_{x_1=0}^{x}\int_{x_2=0}^{x_1}\int_{x_3=0}^{x_2}\int_{x_4=0}^{x_3} G(x-x_1)G(x_1-x_2) \times G(x_2-x_3)G(x_3-x_4)G'(x_4) \times K_4(x_1,x_2,x_3,x_4)\,dx_4\,dx_3\,dx_2\,dx_1 \quad (15)$$

$$K_4(x_1,x_2,x_3,x_4) = \langle \varepsilon(x_1)\varepsilon(x_2)\varepsilon(x_3)\varepsilon(x_4)\rangle \quad (16)$$

By assumption $\varepsilon(x)$ is a normal random function, hence the term in equation (16) has the following correlation structure

$$K_4(x_1,\ldots,x_4) = K(x_1-x_2)K(x_3-x_4) + K(x_1-x_3)K(x_2-x_4) + K(x_1-x_4)K(x_2-x_3) \quad (17)$$

Explicit forms of equations (12) and (14), for the exponential correlation function, are given in Appendix A.

4. Bourret's approximation

Substituting equation (7) into equation (2) and employing equation (3) yields

$$\frac{d^2w}{dx^2} + w - \int_{x_1=0}^{x} G(x-x_1)\varepsilon(x)\varepsilon(x_1)w(x_1)\,dx_1 = 0 \quad (18)$$

Statistical averaging of equation (18) and making use of equation (8) results in the equation for the approximate average solution

$$\frac{d^2\langle w\rangle}{dx^2} + \langle w\rangle + \int_{x_1=0}^{x} G(x-x_1) \times K(x-x_1)\langle w(x_1)\rangle\,dx_1 = 0 \quad (19)$$

Equation (19) can be solved by the one-sided Laplace transform defined as

$$L[f(x)] = \hat{f}(p) = \int_{x=0}^{\infty} \exp(-px)f(x)\,dx \quad (20)$$

The solution of equation (19) in the Laplace transform domain can be written in the form

$$\langle \hat{w}(p)\rangle = \frac{p}{p^2+1-L[G(x)K(x)]} \quad (21)$$

To obtain the inverse Laplace transform of equation (21) in an explicit form one should specify the form of the correlation function $K(x)$. Specifying equation (6a) for $n=1$ one obtains the exponential correlation function

$$K(\tau) = s^2\exp(-b|\tau|) \quad (22)$$

which is the common choice in wave propagation problems. In equation (22) s^2 and b are the dimensionless variance and correlation length, respectively. It should be noted that s may be treated here as a coefficient of variation of the random medium.[3,10]

From equation (21), by the use of Laplace transform and the substitution of equation (22), one obtains

$$\langle \hat{w}(p)\rangle = U_3(p)/U_4(p) \quad (23)$$

where

$$U_3(p) = p^3 + 2bp^2 + (1+b^2)p \quad (24a)$$

$$U_4(p) = p^4 + 2bp^3 + (b^2+2)p^2 + 2bp + b^2 - s^2 + 1 \quad (24b)$$

The inverse Laplace transform of equation (23) yields

$$\langle w(x)\rangle = \mathrm{Re}\sum_{n=1}^{4} [U_3(p_n)/U_4'(p_n)]\exp(p_n x) \quad (25)$$

where

$$U_4'(p) = dU_4(p)/dp \quad (26)$$

and p_n, $(n=1,2,3,4)$, are the roots of the polynomial (24b). An explicit form of the roots p_n are given in Appendix A.

5. Range of validity and error of approximate average solution

In the present paper the mean square convergence is considered. The condition for this kind of convergence of the solution series (9) is

$$\lim_{n\to\infty}\langle [e_w^{(n)} - 0]^2\rangle = 0 \quad (27)$$

where $e_w^{(n)}$ is the absolute error associated with approximation

$$w^{(n)}(x) = w_0(x) + \sum_{j=1}^{n} w_j(x), \qquad x \geq 0 \tag{28}$$

The absolute error is defined as

$$e_w^{(n)} = \|w(x) - w^{(n)}(x)\| = \left\| \sum_{j=n+1}^{\infty} w_j(x) \right\| \tag{29}$$

where $\|\cdot\|$ is the maximum over $x \geq 0$ of the absolute value of the operand.

Equation (27) can be written as

$$\lim_{n\to\infty}\langle [e_w^{(n)}]^2 \rangle = \lim_{n\to\infty} A^{(n)} = 0 \tag{30}$$

where

$$A^{(n)} = \left\| \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} \langle w_i(x) w_j(x) \rangle \right\| \tag{31}$$

In Appendix B it is shown that $A^{(n)}$ is bounded by the expression

$$A^{(n)} \leq [(sx)^2 - sx + 1]\exp(sx) - 1 - \sum_{m=2}^{2n+1} \frac{(m-1)^2}{m!}(sx)^m \tag{32}$$

The relation (32) shows that the limit $\lim_{n\to\infty} A^{(n)} = 0$ is satisfied independently on the value of $sx \geq 0$. The derivation of the bound for $A^{(n)}$ is based on the relation

$$\|\rho(x)\| = \|K(x)/s^2\| \leq 1 \tag{33}$$

where $K(x)$ is the correlation function of $\varepsilon(x)$.

Utilizing the form of the correlation structure of normal random function (Appendix B) yields

$$\langle w(x) \rangle = w_0(x) + \sum_{m=1}^{\infty} \langle w_{2m}(x) \rangle \tag{34}$$

and nth order approximate average solution equals

$$\langle w(x) \rangle \approx \langle w^{(n)}(x) \rangle = w_0(x) + \sum_{m=1}^{n} \langle w_{2m}(x) \rangle \tag{35}$$

The relation (32) was derived as the requirement for mean square convergence of the solution series (9). Using a similar technique (Appendix B) one obtains the relation for the error associated with various order approximations of the average solution, equation (35), as

$$e_w^{(2n)} = \|\langle w(x) \rangle - \langle w^{(2n)}(x) \rangle\| \leq (sx - 1)\exp(sx) + 1 - \sum_{m=2}^{2n+1} \frac{m-1}{m!}(sx)^m \tag{36}$$

Equation (36) represents an upper bound on the errors. This equation is a sufficient but probably not necessary condition for the range of convergence of the average solution.

It is easy to check that when keeping the accuracy requirement at the 5% level the second-order approximation (equation [A2]) is valid, according to equation (36), for $0 \leq sx < 0.75$. The fourth-order approximation (equation [A3]) extends the validity up to $sx < 1.35$.

It should be noted that in the wave propagation problems related to dynamic soil-foundation interaction we are often interested in the small distances from the foundation (small range of space variable x).

6. Numerical results and conclusions

The effect of the randomness of the medium through which the wave propagates, i.e., increasing the damping due to the scattering in the elastic random medium, is displayed in *Figures 1* and *2* for correlation length $b = 1$. These plots compare the results (equations [A2], [A3], and [25]) obtained on the basis of two different methods: Adomian's decomposition and Bourret's approximation, respectively. The solutions are exact and equal in some range of x. This range depends on the parameters of the correlation functions s and b. From the study of *Figures 1* and *2* it is seen that increasing the order of approximation in equation (35) improves the solution, and it tends to the solution obtained via Bourret's approximation. A similar procedure as for the average displacement, $\langle w(x) \rangle$, can be used to obtain the variance function of the displacement

$$V_w(x) = \langle [w(x) - \langle w(x) \rangle]^2 \rangle \tag{37}$$

and in the derivation of the formula for the error associated with various order approximations of the variance function.

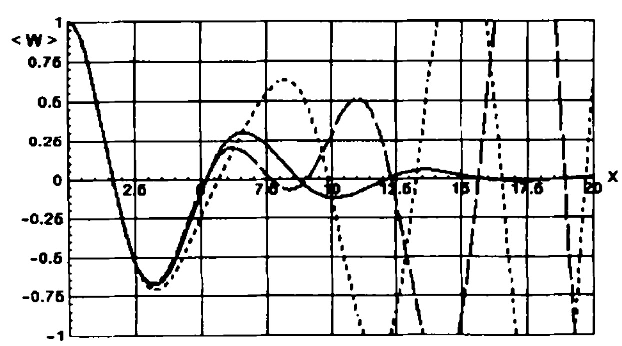

Figure 1. Comparison of the approximate average solution for $b=1$, $s=0.9$: equation (A2) ······; equation (A3) ------; and equation (25) ——.

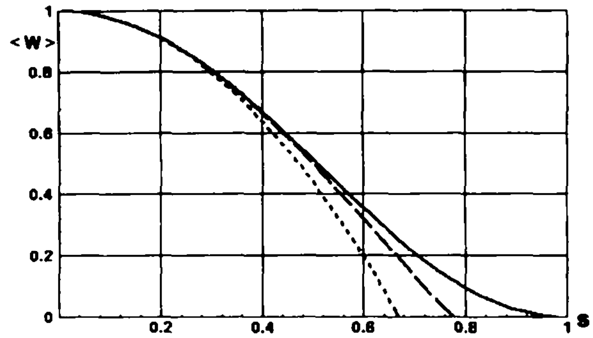

Figure 2. Influence of the coefficient of variation s on the approximate average solution for $b=1$, $x=4\pi$: equation (A2) ······; equation (A3) - - - - - -; and equation (25) ——.

Combining equations (11), (17), and (37) leads to the second-order approximation of the variance function as

$$V_w(x) \cong \langle w_1^2(x)\rangle + \langle w_2^2(x)\rangle - \langle w_2(x)\rangle^2 \qquad (38)$$

The average and variance properties of the solution are governed by two dimensionless parameters: Coefficient of variation s and correlation length b.

Higher-order terms in the expansion for $\langle w(x)\rangle$ and $V_w(x)$, obtained via Adomian's decomposition, become increasingly complex and are difficult to study parametrically.

Therefore it is easy to see that Bourret's approximation leads to simplification of the problem. This approximation may be used to considering several interesting physical problems. Two simple models related to the dynamic soil-foundation interaction can be mentioned here:

1. Dynamically loaded semi-infinite rod resting on a random elastic foundation[8,9,11],
2. Dynamically loaded conical rod, in shear[10] and in torsion,[12] with random shear modulus.

Case 1 could be regarded as a model of a pile embedded in soil or as a potential crude model to estimate the dynamic-stiffness coefficient of a rigid foundation on a half space's surface.

These models can indeed be used to discuss the vital aspects of wave propagation toward infinity. However the first case should not be used to model an actual site, unlike the conical rods in shear and in torsion.

The above analysis indicates that

1. The Adomian's decomposition method leads to the solution with the assumed level of accuracy and allows the easier computation of $\langle w(x)\rangle$ and $V_w(x)$ for low values of x and s.
2. The solution obtained via Bourret's approximation allows, in a more efficient way, the computation of the first moment, especially for higher values of x and s.
3. From the computational point of view one may use the Adomian's decomposition to obtain a near displacement field, and the Bourret's approximation for the far field.

Appendix A

The one-sided Green's function for the deterministic operator $d^2/dx^2 + 1$ equals

$$G(x) = \sin x \qquad (A1)$$

In this paper, to derive an explicit solution, we employ the exponential correlation function (equation 22). Hence the second-order approximation for the average solution, according to equation (12), can be written as

$$\begin{aligned}\langle w(x)\rangle &\approx \langle w^{(2)}(x)\rangle \\ &= \cos x + s^2\int_{x_1=0}^{x}\int_{x_2=0}^{x_1}\sin(x-x_1) \\ &\quad\times \sin(x_1-x_2)\cos x_2 \\ &\quad\times \exp(-b(x_1-x_2))\,dx_2\,dx_1 \qquad (A2)\end{aligned}$$

The fourth-order approximation for the average solution, according to equation (14), assumes the form

$$\langle w^{(4)}(x)\rangle = \langle w^{(2)}(x)\rangle + \langle w_4(x)\rangle \qquad (A3)$$

where

$$\begin{aligned}\langle w_4(x)\rangle = s^4 &\int_{x_1=0}^{x}\int_{x_2=0}^{x_1}\int_{x_3=0}^{x_2}\int_{x_4=0}^{x_3}\sin(x-x_1) \\ &\times \sin(x_1-x_2)\sin(x_2-x_3)\sin(x_3-x_4) \\ &\times \cos x_4[\exp(-b(x_1-x_2+x_3-x_4)) \\ &+ 2\exp(-b(x_1-x_3+x_2-x_4))] \\ &\times dx_4\,dx_3\,dx_2\,dx_1 \qquad (A4)\end{aligned}$$

In the case of Bourret approximation the roots p_n in equation (25) can be expressed in explicit form as

$$p_{1,2} = \frac{1}{2}\left\{-b + [b^2 - 4 \pm 4(s^2-b^2)^{1/2}]^{1/2}\right\} \qquad (A5)$$

$$p_{3,4} = \frac{1}{2}\left\{-b - [b^2 - 4 \pm 4(s^2-b^2)^{1/2}]^{1/2}\right\} \qquad (A6)$$

Appendix B

The expression for $A^{(n)}$, equation (31), implies

$$0 \leqslant A^{(n)} \leqslant \sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty}\|\langle w_i(x)w_j(x)\rangle\| \qquad (B1)$$

where

$$\langle w_i(x)w_j(x)\rangle$$
$$= (-1)^{i+j}\int_{x_1=0}^{x}\cdots\int_{x_{i-0}}^{x_{i-1}}\int_{x_{i+1}=0}^{x_i}\cdots$$

$$\int_{x_{i+j}=0}^{x_{i+j-1}} [G(x-x_1)\cdots G(x_{i-1}-x_i)]$$
$$\times G'(x_i)\big[G(x_i-x_{i+1})\cdots$$
$$G(x_{i+j-1}-x_{i+j})\big]G'(x_{i+j})$$
$$\times\big[\varepsilon(x_1)\cdots\varepsilon(x_{i+j})\big][dx_{i+j}\dots dx_1] \qquad \text{(B2)}$$

An assumption that $\varepsilon(x)$ is a zero mean normal random function leads to the following correlation structure

$$\langle\varepsilon(x_1)\dots\varepsilon(x_{i+j})\rangle = D_{i+j}s^{i+j}\sum_{m=1}^{i+j-1}\alpha_{i+j,m} \qquad \text{(B3)}$$

where

$$D_{i+j} = \begin{cases} 1, & i+j = even \\ 0, & i+j = odd \end{cases} \qquad \text{(B4)}$$

Each term $\alpha_{i+j,m}$, $(m = 1,2,\dots,i+j-1)$, is a product of $(i+j-1)/2$ normalized correlation functions

$$\rho(x_m - x_n) = K(x_m - x_n)/s^2 \qquad \text{(B5)}$$

evaluated at all the possible combinations $(x_m - x_n)$ of $(x_1,\dots,x_{i+j})$. For $i+j=2$ we have

$$\langle\varepsilon(x_1)\varepsilon(x_2)\rangle = s^2\alpha_{2,1} = s^2\rho(x_1-x_2) \qquad \text{(B6)}$$

and for $i+j=4$

$$\langle\varepsilon(x_1)\varepsilon(x_2)\varepsilon(x_3)\varepsilon(x_4)\rangle = s^4(\alpha_{4,1}+\alpha_{4,2}+\alpha_{4,3}) \qquad \text{(B7)}$$

where

$$\alpha_{4,1} = \rho(x_1-x_2)\rho(x_3-x_4),$$
$$\alpha_{4,2} = \rho(x_1-x_3)\rho(x_2-x_4),$$
$$\alpha_{4,3} = \rho(x_1-x_4)\rho(x_2-x_3) \qquad \text{(B8)}$$

In further considerations we use the relations

$$\|G(x)\| \leqslant 1 \qquad \text{(B9)}$$
$$\|\rho(x)\| \leqslant 1 \qquad \text{(B10)}$$

The relation (B10) implies that

$$\|\alpha_{i+j,m}\| \leqslant 1 \qquad \text{(B11)}$$

Hence,

$$\sum_{m=1}^{i+j-1}\|\alpha_{i+j,m}\| \leqslant i+j-1 \qquad \text{(B12)}$$

Combining equations (B1)–(B12) and taking into account the result of integration

$$\int_{x_1=0}^{x}\cdots\int_{x_{i+j}}^{x_{i+j-1}} dx_{i+j}\dots dx_1 = \frac{x^{i+j}}{(i+j)!} \qquad \text{(B13)}$$

one obtains

$$A^{(n)} \leqslant f_n(sx) = \sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty}\frac{i+j-1}{(i+j)!}(sx)^{i+j} \qquad \text{(B14)}$$

The double series (B14) can be written as

$$f_n(sx) = \sum_{m=2n+2}^{\infty}\frac{(m-1)^2}{m!}(sx)^m$$
$$= g(sx) - \sum_{m=2}^{2n+1}\frac{(m-1)^2}{m!}(sx)^m \qquad \text{(B15)}$$

where

$$g(sx) = \sum_{m=2}^{\infty}\frac{(m-1)^2}{m!}(sx)^m \qquad \text{(B16)}$$

To evaluate the sum of the series (B16) one can decompose it as

$$g(sx) = \sum_{m=0}^{\infty}\frac{m^2}{m!}(sx)^m - 2\sum_{m=0}^{\infty}\frac{m}{m!}(sx)^m$$
$$+ \sum_{m=0}^{\infty}\frac{1}{m!}(sx)^m - 1 \qquad \text{(B17)}$$

Hence the sum (B15) equals

$$f_n(sx) = [(sx)^2 - sx + 1]\exp(sx) - 1$$
$$- \sum_{m=2}^{2n+1}\frac{(m-1)^2}{m!}(sx)^m,$$
$$(n = 1,2,3,\dots) \qquad \text{(B18)}$$

Estimation of the error of approximate average solution (equation [36]) is based on the above relations and uses a similar procedure. Using equation (35) one obtains

$$e_w^{(2n)} = \|\langle w(x)\rangle - \langle w^{(2n)}(x)\rangle\|$$
$$\leqslant \left\|\sum_{m=2n+2}^{\infty}\langle w_m(x)\rangle\right\|$$
$$\leqslant \sum_{m=1}^{\infty}\|\langle w_{2m+2}(x)\rangle\| \qquad \text{(B19)}$$

Combining equations (B3), (B9)–(B12), and (B19) yields

$$e_w^{(2n)} \leqslant \sum_{m=2n+2}^{\infty}\frac{m-1}{m!}(sx)^m \qquad \text{(B20)}$$

It is convenient to write relation (B20) in the form

$$e_w^{(2n)} \leqslant \sum_{m=2}^{\infty} \frac{m-1}{m!}(sx)^m - \sum_{m=2}^{2n+1} \frac{m-1}{m!}(sx)^m, \qquad (n = 1, 2, 3, \ldots) \tag{B21}$$

which leads to relation (36).

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